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Inverse heat conduction problem of determining time-dependent heat transfer coefficient

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Abstract

The time-dependent Biot number in a one-dimensional linear heat conduction problem is obtained from the solutions of the inverse heat conduction problems of determining boundary heat flux and boundary temperature. The sequential function specification method with the linear basis function and the assumption of linearly varying future boundary heat flux or temperature components is used to solve the inverse problem. The expression for Biot number is found to be a nonlinear function of measured temperatures. The variance in input data is shown to cause variance and nonlinear bias in estimated Biot number. The method presented offers three tunable parameters that may be used to improve the quality of the solution.

1. Introduction

A heat conduction problem in a solid with the initial and boundary conditions completely specified is a well-posed problem that can be solved by various analytical and numerical methods. On the other hand, when the boundary condition is to be determined from temperature measurement data inside the solid, the problem is an ill-posed one known as the inverse heat conduction problem (IHCP) [1]. Although the analytical solution of IHCP exists for the case of one-dimensional problem [2], a numerical method is generally preferable since it offers control over the accuracy and the stability of the solution. Among the well-known numerical methods are the space-marching technique [3], the frequency domain adjoint method [4], the mollification method [5], the iterative regularization method [6], the direct sensitivity coefficient method [7], and the sequential function specification method [1]. The aims of these methods are to obtain a solution that is accurate and not very sensitive to changes in input temperature data.

Most of the inverse heat conduction problems that have been investigated so far are concerned with the estimation of boundary heat flux. Another interesting problem that has not yet received as much interest is the estimation of heat transfer coefficient. Osman and Beck [8] treated the problem of estimating time-dependent heat transfer coefficient in the quenching of a sphere as a nonlinear parameter estimation problem. Heat transfer coefficient was assumed to be a piecewise constant function of time. The unknown heat transfer coefficient parameters were estimated one by one using the sequential function specification method. Naylor and Oosthuizen [9] employed the temperature-time data measured at subsurface locations to determine the heat transfer coefficient in a forced convective flow over a square prism. They expressed the distribution of heat transfer coefficient in terms of several piecewise constant functions. The coefficients were computed using an iterative algorithm. Hernandez-Morales et al.

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Nomenclature

A	response vector
B	Biot number
C	response vector
c_p	heat capacity
D	coefficient matrix
$E(y)$	expected value of random variable y
f	probability density distribution function
h	heat transfer coefficient
L	length
l, m	dummy indices
n	the number of Biot number components to be determined
p	ratio of time step for estimated Biot number components to time step for temperature measurements
q	boundary heat flux
r	future-time parameter
S	transformation matrix
T	temperature
T_0	initial temperature
T_∞	ambient temperature
t	time
$\mathrm{Var}(y)$	variance of random variable y
x	location
x_0	sensor location.

Greek symbols

$\alpha_j^{(i)}$	coefficient that relates estimated boundary heat flux component to measured temperature
$\beta_j^{(i)}$	coefficient that relates estimated boundary temperature to measured temperature
Δ_d	deterministic bias
Δ_n	nonlinear bias
ε	temperature measurement error
ψ	response function
κ	thermal conductivity
θ	boundary temperature
ρ	density
σ^2	variance of temperature measurements
ψ	response function.

Subscripts and superscripts

i, j, k, l, m	indices.

[10] studied the one-dimensional problem of estimating the transient heat transfer coefficient at the surface of steel bars subjected to quenching using the sequential function specification method. They found that filtering the input data led to improved estimation. Mehrotra et al. [11] estimated interfacial heat transfer coefficient in solidification of a molten metal on a metal substrate. Their transient one-dimensional problem was divided into a direct region and an indirect region. The solution for the direct region was obtained using a conventional method. The Burggraf solution [2] was then used to compute the temperature and heat flux at the interface between the molten metal and the substrate, from which the heat transfer coefficient could be determined in a straightforward manner. Xu and Chen [12] studied the steady-state nonlinear problem of determining the heat transfer coefficient in two-phase mixture flow in an inclined tube. Their algorithm was a simple iterative procedure. Most recently, Martin and Dulikravich [13] employed the boundary element method to set up the inverse problem of determining boundary heat flux and boundary temperature simultaneously in a steady-state multidimensional problem. The single value decomposition method was then used to obtain stabilized solutions for boundary heat flux and boundary temperature, from which heat transfer coefficient was determined.

In this paper, an algorithm for estimating time-dependent heat transfer coefficient for a one-dimensional linear inverse heat conduction problem is proposed. The method used is the sequential function specification method with the linear basis function and the assumption of linearly varying future boundary heat flux (or temperature) components. Recent results by Chantasiriwan [14] showed that this method yielded better estimates of boundary condition than the well-known sequential function specification method [1]. Hence, it is expected that the estimation of heat transfer coefficient should perform better with the new method as well. The following sections will describe the matrix formulation of the algorithm, which will facilitate computer implementation. The method for analyzing the accuracy and stability of the estimate will then be described. Sample results and discussion of how to improve the estimate will follow. Finally, the conclusions that can be drawn from this paper will be given.

2. Mathematical formulation of the problem

The problem to be considered is shown in Fig. 1. A one-dimensional object is subjected to unknown time-dependent heat transfer coefficient at one end whereas

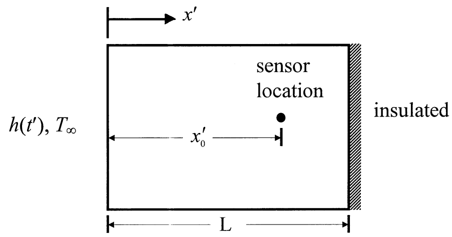

Fig. 1. One-dimensional inverse heat conduction problem to be solved for $h(t')$.

the other end is insulated. The ambient temperature is assumed constant throughout the time period considered. The temperature measurement is made at a distance from the boundary of unknown heat transfer coefficient. The measurement data along with the known geometrical and thermophysical data give rise to the inverse heat conduction problem, which can be mathematically described by the following governing equation, initial condition, and boundary condition.

$$\rho c_p \frac{\partial T'(x',t')}{\partial t'} = \kappa \frac{\partial^2 T'(x',t')}{\partial x'^2} \quad (1)$$

$$T'(x',0) = T_0 \quad (2)$$

$$\left.\frac{\partial T'(x',t')}{\partial x'}\right|_{x'=L} = 0 \quad (3)$$

Temperature measurements at sensor location x'_0, which is between 0 and L, are available at a regular time interval.

$$T'(x'_0, i\Delta t') = T'_i \quad (4)$$

The heat transfer coefficient at the convective boundary $x'=0$ is to be determined. The definition of $h(t')$ is given by

$$-\kappa \left.\frac{\partial T'(x',t')}{\partial x'}\right|_{x'=0} = h(t')(T_\infty - T') \quad (5)$$

Define dimensionless variables $x = x'/L$, $t = \kappa t'/\rho c_p L^2$, $T = (T' - T_0)/(T_\infty - T_0)$, and $B(t) = h(t')L/\kappa$. Eqs. (1)–(5) can be rewritten in dimensionless forms.

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \quad (6)$$

$$T(x,0) = 0 \quad (7)$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=1} = 0 \quad (8)$$

$$T(x_0, i\Delta t) = T_i \quad (9)$$

$$-\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = B(t)(1-T) \quad (10)$$

The goal of the IHCP solution is to determine $B(t)$. Instead of devising an algorithm to determine $B(t)$ directly, it is more convenient to estimate boundary heat flux $q(t)$ and boundary temperature $\theta(t)$ at $x = 0$ from Eqs. (6)–(9), and use them to obtain the expression for $B(t)$.

3. Determination of boundary heat flux

The expression for boundary heat flux into the object is

$$q(t) = -\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} \quad (11)$$

Chantasiriwan [14] described a sequential function specification algorithm for estimating $q(t)$. The algorithm makes use of piecewise linear function in estimating boundary heat flux component by component. The solution is stabilized by employing future-time temperature measurements as input data, and assuming that heat flux components vary linearly. A distinctive feature of that algorithm and the conventional sequential function specification algorithm [1] is the presence of future-time parameter r, which is related to the number of future-time measurements used as input data. This parameter acts as a stabilizing parameter in that the solution becomes more stable as r increases. However, that algorithm is limited to cases in which the time step of temperature measurements equals the time step of the estimated heat flux components. Because more input data will probably lead to a more stable solution, it may be advantageous to allow the former to be smaller than the latter. In the algorithm to be used in this paper, the time step of temperature measurement is Δt, and the time step of the estimated heat flux components is $p\Delta t$, where p is a positive integer. The revised formulation for the sequential function specification algorithm will now be described.

The 'current' heat flux component, $q^{(i)}$ $(1 \le i \le n)$, is estimated using rp 'future' temperature measurements, $T_{(i-1)p+1}$, $T_{(i-1)p+2}$, ... ,$T_{(i+r-1)p}$. It is assumed that the basis function of the boundary heat flux components is the piecewise linear function and that future heat flux components vary linearly (see Fig. 2.) For $i-1 \le k \le i+r-2$ and $1 \le m \le p$, temperatures at x_0 are related to boundary heat flux components as follows.

$$
\begin{aligned}
T_{kp+m} = {} & \sum_{j=1}^{i-1} \frac{q^{(j)}}{p\Delta t}[\phi(x_0, \\
& ((k-j+1)p+m)\Delta t) - 2\phi(x_0, \\
& ((k-j)p+m)\Delta t) + \phi(x_0, \\
& ((k-j-1)p+m)\Delta t)] + \sum_{j=0}^{k-i+1} \frac{q^{(i+j)}}{p\Delta t}[\phi(x_0, \\
& ((k-i-j+1)p+m)\Delta t) - 2\phi(x_0, \\
& ((k-i-j)p+m)\Delta t) + \phi(x_0, \\
& ((k-i-j-1)p+m)\Delta t)]
\end{aligned} \tag{12}
$$

where

$$
\psi(x_0, t) = \begin{cases} \dfrac{t^2}{2} + t\left(\dfrac{x^2}{2} - x_0 + \dfrac{1}{3}\right) - 2\displaystyle\sum_{m=1}^{\infty} \frac{\cos(m\pi x_0)}{(m\pi)^4}\left(1 - e^{-m^2\pi^2 t}\right), & \text{for} \quad t \ge 0 \\ 0, & \text{for} \quad t < 0 \end{cases} \tag{13}
$$

is the temperature response at $x = x_0$ to the heat conduction problem described by Eqs. (6)–(8) and the condition that the linearly increasing heat flux having unity slope is applied at $x = 0$. Since it is assumed that future heat flux components $q^{(i+1)}$, $q^{(i+2)}$, …, $q^{(i+r-1)}$ vary linearly, $q^{(i+l)}$ can be expressed in terms of $q^{(i)}$ and $q^{(i-1)}$:

$$
q^{(i+l)} = (l+1)q^{(i)} - lq^{(i-1)} \tag{14}
$$

for $1 \le l \le r-1$. Now, substitute Eq. (14) into (12), and simplify the result.

$$
\begin{aligned}
T_{kp+m} = {} & \frac{q^{(i)}}{p\Delta t}\phi(x_0, \\
& ((k-j+1)p+m)\Delta t) + \frac{q^{(i-1)}}{p\Delta t}[\phi(x_0, \\
& ((k-i+2)p+m)\Delta t) - 2\phi(x_0, \\
& ((k-i+1)p+m)\Delta t)] + \sum_{j=1}^{i-2} \frac{q^{(j)}}{p\Delta t}[\phi(x_0, \\
& ((k-j+1)p+m)\Delta t) - 2\phi(x_0, \\
& ((k-j)p+m)\Delta t) + \phi(x_0, ((k-j-1)p+m)\Delta t)]
\end{aligned} \tag{15}
$$

Because $q^{(i-1)}$, $q^{(i-2)}$, …, $q^{(1)}$ are known from previous calculations, Eq. (15) represents an overdetermined system of linear algebraic equations with $q^{(i)}$ as the only unknown for the current calculation. Let us define the following vectors and matrices:

$$
\mathbf{T} = [T_1 \quad T_2 \quad \ldots \quad T_{(n+r-1)p}]^{\mathrm{T}}
$$

$$
\mathbf{A} = \frac{1}{p\Delta t}[\phi(x_0, \Delta t) \quad \phi(x_0, 2\Delta t) \quad \ldots \quad \phi(x_0, rp\Delta t)]^{\mathrm{T}}
$$

$$
\mathbf{C}^{(1)} = \frac{1}{p\Delta t}\begin{bmatrix} \phi(x_0, (p+1)\Delta t) - 2\phi(x_0, \Delta t) \\ \phi(x_0, (p+2)\Delta t) - 2\phi(x_0, 2\Delta t) \\ \vdots \\ \phi(x_0, (r+1)p\Delta t) - 2\phi(x_0, rp\Delta t) \end{bmatrix}
$$

$$
\mathbf{C}^{(i)} = \frac{1}{p\Delta t}\begin{bmatrix} \phi(x_0, (ip+1)\Delta t) - 2\phi(x_0, ((i-1)p+1)\Delta t) + \phi(x_0, ((i-2)p+1)\Delta t) \\ \phi(x_0, (ip+2)\Delta t) - 2\phi(x_0, ((i-1)p+2)\Delta t) + \phi(x_0, ((i-2)p+2)\Delta t) \\ \vdots \\ \phi(x_0, (i+r)p\Delta t) - 2\phi(x_0, (i+r-1)p\Delta t) + \phi(x_0, (i+r-2)p\Delta t) \end{bmatrix}
$$

for $2 \le i \le n-1$, and

$$
\mathbf{S}^{(i)} = \left.\left[\begin{array}{cccc|cccc|cccc} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots 0 & & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad 0 \quad \ldots \quad 0}_{(i-1)p \text{ columns}} & & & & \underbrace{0 \quad 0 \quad \ldots \quad 1}_{rp \text{ columns}} & & & & \underbrace{0 \quad 0 \quad \ldots \quad 0}_{(n-i)p \text{ columns}} & & & \end{array}\right]\right\} rp \text{ rows}
$$

for $1 \leq i \leq n$. Eq. (15) can now be rewritten as a matrix equation:

$$\mathbf{S}^{(i)}\mathbf{T} = \mathbf{A}q^{(i)} + \sum_{k=1}^{i-1}\mathbf{C}^{(i-k)}q^{(k)} \tag{16}$$

It is useful to express the unknown $q^{(i)}$ in terms of all other quantities.

$$\theta(t) = T(0, t) \tag{21}$$

Let $\psi(x_0, t)$ be the temperature response at $x = x_0$ to the heat conduction problem described by Eqs. (6)–(8) and the condition that the linearly increasing temperature having unity slope is applied at $x = 0$. The expression for $\psi(x_0, t)$ is

$$\psi(x_0, t) = \begin{cases} t - 2\sum_{m=1}^{\infty}\dfrac{\sin((m-0.5)\pi x_0)}{((m-0.5)\pi)^3}(1 - e^{-(m-0.5)^2\pi^2 t}), & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \tag{22}$$

$$q^{(i)} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\left[\mathbf{S}^{(i)}\mathbf{T} - \sum_{k=1}^{i-1}\mathbf{C}^{(i-k)}g^{(k)}\right] \tag{17}$$

Furthermore, let

$$q^{(i)} = \mathbf{D}^{(i)}\mathbf{T} \tag{18}$$

The coefficient matrix $\mathbf{D}^{(i)}$, which relates the unknown heat flux component $q^{(i)}$ to known temperature measurement data, $T_1, T_2, \ldots, T_{(n+r-1)p}$, can be found from Eqs. (17) and (18).

$$\mathbf{D}^{(i)} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\left[\mathbf{S}^{(i)} - \sum_{k=1}^{i-1}\mathbf{C}^{(i-k)}\mathbf{D}^{(k)}\right] \tag{19}$$

Knowledge of $\mathbf{D}^{(i)}$ allows us to write $q^{(i)}$ in terms of $T_1, T_2, \ldots, T_{(n+r-1)p}$.

$$q^{(i)} = \sum_{k=1}^{(n+r-1)p}\alpha_k^{(i)}T_k \tag{20}$$

4. Determination of boundary temperature

The expression for boundary temperature is

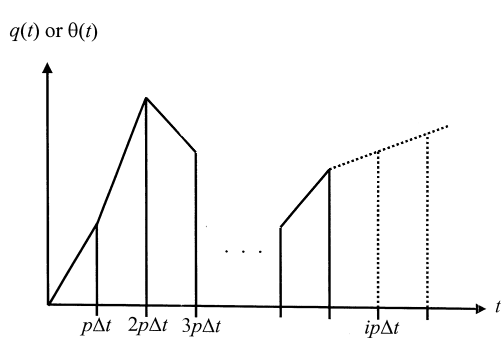

Fig. 2. Pictorial representation of the sequential function specification algorithm.

With the replacement of $\phi(x_0, t)$ by $\psi(x_0, t)$ in the expressions for $\mathbf{A}$ and $\mathbf{C}^{(i)}$, the procedure described previously can be used to determine boundary temperature $\theta^{(i)}$ as a function of temperature measurement data.

$$\theta^{(i)} = \sum_{k=1}^{(n+r-1)p}\beta_k^{(i)}T_k \tag{23}$$

5. Determination of time-dependent Biot number

Once $q^{(i)}$ and $\theta^{(i)}$ have been determined, $B^{(i)}$ can be obtained from

$$B^{(i)} = \frac{q^{(i)}}{1 - \theta^{(i)}} \tag{24}$$

It is interesting to note that $B^{(i)}$ is a nonlinear function of measured temperatures,

$$B^{(i)} = \frac{\sum_{k=1}^{(n+r-1)p}\alpha_k^{(i)}T_k}{1 - \sum_{k=1}^{(n+r-1)p}\beta_k^{(i)}T_k} \tag{25}$$

whereas $q^{(i)}$ and $\theta^{(i)}$ are linear functions of temperatures. Hence, the evaluation of statistical errors in $B^{(i)}$, resulting from errors in temperature measurement, is more complicated than the evaluation of statistical errors in $q^{(i)}$ and $\theta^{(i)}$.

It is useful to make the following statistical assumptions regarding temperature measurement errors [15].

1. Additive errors: $T_j = \bar{T}_j + \varepsilon_j$
2. Zero mean errors: $\mathrm{E}(\varepsilon_j) = 0$
3. Constant variance: $\mathrm{Var}(\varepsilon_j) = \sigma^2$
4. Uncorrelated errors: $\mathrm{E}(\varepsilon_j\varepsilon_k) = 0$ if $j \neq k$
5. Normal probability distribution for errors
6. Nonstochastic independent variable

As a result of these assumptions, the probability density function for errors is

$$f(\varepsilon_j) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\varepsilon_j}{\sigma}\right)^2 \right], \quad -\infty < \varepsilon_j < \infty \tag{26}$$

In inverse heat conduction problem of estimating boundary heat flux or boundary temperature, the quality of the solution is determined by two measures: the deterministic bias and the variance of the solution. Deterministic bias represents the difference between the estimated solution and exact solution when temperature measurements are error-free. The deterministic biases for $q^{(i)}$ and $\theta^{(i)}$ may be defined, respectively, as

$$\Delta_{d,\,q} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(q(ip\Delta t) - \mathrm{E}(q^{(i)}))^2} \tag{27}$$

and

$$\Delta_{d,\,\theta} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\theta(ip\Delta t) - \mathrm{E}(\theta^{(i)}))^2} \tag{28}$$

where

$$\mathrm{E}(q^{(i)}) = \sum_{k=1}^{(n+r-1)p} \alpha_k^{(i)} \bar{T}_k \tag{29}$$

and

$$\mathrm{E}(\theta^{(i)}) = \sum_{k=1}^{(n+r-1)p} \beta_k^{(i)} \bar{T}_k \tag{30}$$

are the expected values of the estimated boundary heat flux component and the estimated boundary temperature component. Thus, the deterministic bias depends on the solution algorithm, but not the statistical errors present in actual input data. On the other hand, the variance of the solution is a function of the variance of input data. As a consequence of the above assumptions regarding temperature measurement errors, the variances of boundary heat flux and boundary temperature are, respectively,

$$\mathrm{Var}(q^{(i)}) = \sigma^2 \sum_{k=1}^{(n+r-1)p} (\alpha_k^{(i)})^2 \tag{31}$$

and

$$\mathrm{Var}(\theta^{(i)}) = \sigma^2 \sum_{k=1}^{(n+r-1)p} (\beta_k^{(i)})^2 \tag{32}$$

6. Statistical errors in estimated Biot number

Define average Biot number component as

$$\bar{B}^{(i)} = \frac{\sum_{k=1}^{(n+r-1)p} \alpha_k^{(i)} \bar{T}_k}{1 - \sum_{k=1}^{(n+r-1)p} \beta_k^{(i)} \bar{T}_k} \tag{33}$$

Taylor series expansion for $B^{(i)}$ yields

$$\begin{aligned} B^{(i)} = {} & \bar{B}^{(i)} + \sum_j \frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j} \varepsilon_j + \frac{1}{2}\sum_{j,\,k} \frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\bar{T}_j,\,\bar{T}_k} \varepsilon_j\varepsilon_k \\ & + \frac{1}{6}\sum_{j,\,k,\,l} \frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\bigg|_{\bar{T}_j,\,\bar{T}_k,\,\bar{T}_l} \varepsilon_j\varepsilon_k\varepsilon_l \\ & + \frac{1}{24}\sum_{j,\,k,\,l,\,m} \frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\bigg|_{\bar{T}_j,\,\bar{T}_k,\,\bar{T}_l,\,\bar{T}_m} \varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m \\ & + \cdots \end{aligned} \tag{34}$$

where each index in above summations and summations to follow runs from 1 to $(n+r-1)p$. The expressions for derivatives of $B^{(i)}$ are given below.

$$\frac{\partial B^{(i)}}{\partial T_j} = \frac{\alpha_j^{(i)} + B^{(i)}\beta_j^{(i)}}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)} \tag{35}$$

$$\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k} = \frac{\beta_j^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\alpha_j^{(i)} + 2B^{(i)}\beta_j^{(i)}\beta_k^{(i)}}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)^2} \tag{36}$$

$$\begin{aligned} \frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l} = {} & \frac{1}{\left(1 - \sum_k \beta_k^{(i)} T_k\right)^3} [\beta_j^{(i)}\beta_k^{(i)}\alpha_l^{(i)} \\ & + \beta_j^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\beta_l^{(i)}\alpha_j^{(i)} \\ & + \beta_k^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_l^{(i)}\beta_j^{(i)}\alpha_k^{(i)} \\ & + \beta_l^{(i)}\beta_k^{(i)}\alpha_j^{(i)} + 6B^{(i)}\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)}] \end{aligned} \tag{37}$$

$$
\begin{aligned}
\frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m} = {} & \frac{1}{\left(1-\sum_k \beta_k^{(i)} T_k\right)^4}[\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)}\alpha_m^{(i)} \\
& + \beta_j^{(i)}\beta_k^{(i)}\beta_m^{(i)}\alpha_l^{(i)} + \beta_j^{(i)}\beta_l^{(i)}\beta_k^{(i)}\alpha_m^{(i)} + \beta_j^{(i)}\beta_l^{(i)}\beta_m^{(i)}\alpha_k^{(i)} \\
& + \beta_j^{(i)}\beta_m^{(i)}\beta_k^{(i)}\alpha_l^{(i)} + \beta_j^{(i)}\beta_m^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_k^{(i)}\beta_j^{(i)}\beta_l^{(i)}\alpha_m^{(i)} \\
& + \beta_k^{(i)}\beta_j^{(i)}\beta_m^{(i)}\alpha_l^{(i)} + \beta_k^{(i)}\beta_l^{(i)}\beta_j^{(i)}\alpha_m^{(i)} + \beta_k^{(i)}\beta_l^{(i)}\beta_m^{(i)}\alpha_j^{(i)} \\
& + \beta_k^{(i)}\beta_m^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_k^{(i)}\beta_m^{(i)}\beta_l^{(i)}\alpha_j^{(i)} + \beta_l^{(i)}\beta_j^{(i)}\beta_k^{(i)}\alpha_m^{(i)} \\
& + \beta_l^{(i)}\beta_j^{(i)}\beta_m^{(i)}\alpha_k^{(i)} + \beta_l^{(i)}\beta_k^{(i)}\beta_j^{(i)}\alpha_m^{(i)} + \beta_l^{(i)}\beta_k^{(i)}\beta_m^{(i)}\alpha_j^{(i)} \\
& + \beta_l^{(i)}\beta_m^{(i)}\beta_j^{(i)}\alpha_k^{(i)} + \beta_l^{(i)}\beta_m^{(i)}\beta_k^{(i)}\alpha_j^{(i)} + \beta_m^{(i)}\beta_j^{(i)}\beta_k^{(i)}\alpha_l^{(i)} \\
& + \beta_m^{(i)}\beta_j^{(i)}\beta_l^{(i)}\alpha_k^{(i)} + \beta_m^{(i)}\beta_k^{(i)}\beta_j^{(i)}\alpha_l^{(i)} + \beta_m^{(i)}\beta_k^{(i)}\beta_l^{(i)}\alpha_j^{(i)} \\
& + \beta_m^{(i)}\beta_l^{(i)}\beta_j^{(i)}\alpha_k^{(i)} + \beta_m^{(i)}\beta_l^{(i)}\beta_k^{(i)}\alpha_j^{(i)} + 24\beta_j^{(i)}\beta_k^{(i)}\beta_l^{(i)}\beta_m^{(i)}]
\end{aligned} \tag{38}
$$

Expected value for $B^{(i)}$ is given by

$$
\begin{aligned}
\mathrm{E}(B^{(i)}) = {} & \mathrm{E}(\bar{B}^{(i)}) + \sum_j \frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j} \mathrm{E}(\varepsilon_j) \\
& + \frac{1}{2}\sum_{j,k} \frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k} \mathrm{E}(\varepsilon_j\varepsilon_k) \\
& + \frac{1}{6}\sum_{j,k,l} \frac{\partial^3 B^{(i)}}{\partial T_j \partial T_k \partial T_l}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l} \mathrm{E}(\varepsilon_j\varepsilon_k\varepsilon_l) \\
& + \frac{1}{24}\sum_{j,k,l,m} \frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m} \\
& \mathrm{E}(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m) + \cdots
\end{aligned}
$$

Note that, in addition to the first moment of ε_j, which is the zero mean, and the second moment of ε_j, which is the variance, the right hand side contains the third moment and the fourth moment of ε_j. With the density distribution function $f(\varepsilon_j)$ given in Eq. (26), they can be simply evaluated.

$$
\mathrm{E}(\varepsilon_j^3) = \int_{-\infty}^{\infty} \varepsilon_j^3 f(\varepsilon_j)\,\mathrm{d}\varepsilon_j = 0 \tag{40}
$$

$$
\mathrm{E}(\varepsilon_j^4) = \int_{-\infty}^{\infty} \varepsilon_j^4 f(\varepsilon_j)\,\mathrm{d}\varepsilon_j = 3\sigma^4 \tag{41}
$$

Hence, the second term on right hand side of Eq. (39) vanishes. As a consequence of the zero correlation between measurement errors, the fourth term also vanishes. The third and fifth terms can be rewritten as

$$
\begin{aligned}
& \frac{1}{2}\sum_{j,k} \frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k} \mathrm{E}(\varepsilon_j\varepsilon_k) = \frac{1}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j} \mathrm{E}(\varepsilon_j^2) \\
& = \frac{\sigma^2}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}
\end{aligned} \tag{42}
$$

and

$$
\begin{aligned}
& \frac{1}{24}\sum_{j,k,l,m} \frac{\partial^4 B^{(i)}}{\partial T_j \partial T_k \partial T_l \partial T_m}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m} \mathrm{E}(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m) \\
& = \frac{1}{24}\left[3\sum_{j\neq k} \frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} \mathrm{E}(\varepsilon_j^2)\mathrm{E}(\varepsilon_k^2) \right. \\
& \left. + \sum_j \frac{\partial^4 B^{(i)}}{\partial T_j^4}\mathrm{E}(\varepsilon_j^4)\right] + \frac{\sigma^4}{8}\left[\sum_{j\neq k} \frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} \right. \\
& \left. + \sum_j \frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\right]
\end{aligned} \tag{43}
$$

Expected value for $B^{(i)}$ can now be expressed in terms of temperature measurement variance σ^2.

$$
\begin{aligned}
\mathrm{E}(B^{(i)}) = {} & \bar{B}^{(i)} + \frac{\sigma^2}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j} + \frac{\sigma^4}{8}\left[\sum_{j\in k} \frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} \right. \\
& \left. + \sum_j \frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\right] + \mathrm{O}(\sigma^6)
\end{aligned} \tag{44}
$$

It is interesting to note that variance in measurement errors will cause the expected value, $\mathrm{E}(B^{(i)})$, of the estimated Biot number component to deviate from the true average value, $\bar{B}^{(i)}$, of the estimated Biot number component. The difference between $\mathrm{E}(B^{(i)})$ and $\bar{B}^{(i)}$ may be called the nonlinear bias $\Delta_{\mathrm{n}}^{(i)}$. If terms of order $\mathrm{O}(\sigma^6)$ and higher are neglected,

$$
\begin{aligned}
\Delta_{\mathrm{n}}^{(i)} = \mathrm{E}(B^{(i)}) - \bar{B}^{(i)} = {} & \frac{\sigma^2}{2}\sum_j \frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j} + \frac{\sigma^4}{8} \\
& \times \left[\sum_{j\neq k} \frac{\partial^4 B^{(i)}}{\partial T_j^2 \partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} + \sum_j \frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\right]
\end{aligned} \tag{45}
$$

In the estimation of boundary heat flux or boundary temperature, where the dependence on measured temperatures is linear, nonlinear bias is zero. However, in a nonlinear estimation such as the problem considered here, nonlinear bias cannot be ignored unless variance of measurement errors is negligible.

The variance of $B^{(i)}$ can be determined from the following definition.

$$
\mathrm{Var}(B^{(i)}) = \mathrm{E}((B^{(i)})^2) - (\mathrm{E}(B^{(i)}))^2 \tag{46}
$$

The right-hand side of Eq. (46) will now be evaluated term by term.

$$
\begin{aligned}
(B^{(i)})^2 =& (\bar{B}^{(i)})^2 + 2\bar{B}^{(i)}\sum_j \frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\varepsilon_j + \bar{B}^{(i)}\sum_{j,k}\frac{\partial^2 B^{(i)}}{\partial T_j \partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\varepsilon_j\varepsilon_k + \frac{\bar{B}^{(i)}}{3}\sum_{j,k,l}\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k\partial T_l}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l}\varepsilon_j\varepsilon_k\varepsilon_l \\
&+ \frac{\bar{B}^{(i)}}{12}\sum_{j,k,l,m}\frac{\partial^4 B^{(i)}}{\partial T_j\partial T_k\partial T_l\partial T_m}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l,\bar{T}_m}\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m + \left(\sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\varepsilon_j\right)^2 + \frac{1}{4}\left(\sum_{j,k}\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\varepsilon_j\varepsilon_k\right)^2 \\
&+ \left(\sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\varepsilon_j\right)\left(\sum_{j,k}\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\varepsilon_j\varepsilon_k\right) + \frac{1}{3}\left(\sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\varepsilon_j\right)\left(\sum_{j,k,l}\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k\partial T_l}\bigg|_{\bar{T}_j,\bar{T}_k,\bar{T}_l}\varepsilon_j\varepsilon_k\varepsilon_l\right) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}((B^{(i)})^2) =& \mathrm{E}((\bar{B}^{(i)})^2) + \bar{B}^{(i)}\sum_j\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\mathrm{E}(\varepsilon_j^2) + \sum_j\left(\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\right)^2\mathrm{E}(\varepsilon_j^2) + \frac{\bar{B}^{(i)}}{12}\sum_j\frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\mathrm{E}(\varepsilon_j^4) \\
&+ \frac{1}{4}\sum_j\left(\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\right)^2\mathrm{E}(\varepsilon_j^4) + \frac{1}{3}\sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j^3}\bigg|_{\bar{T}_j}\mathrm{E}(\varepsilon_j^4) + \frac{\bar{B}^{(i)}}{2}\sum_{j\neq k}\frac{\partial^4 B^{(i)}}{\partial T_j^2\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k}\mathrm{E}(\varepsilon_j^2)\mathrm{E}(\varepsilon_k^2) \\
&+ \frac{1}{4}\sum_{j\neq k}\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\frac{\partial^2 B^{(i)}}{\partial T_k^2}\bigg|_{\bar{T}_k}\mathrm{E}(\varepsilon_j^2)\mathrm{E}(\varepsilon_k^2) + \frac{1}{2}\sum_{j\neq k}\left(\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\right)^2\mathrm{E}(\varepsilon_j^2)\mathrm{E}(\varepsilon_k^2) \\
&+ \sum_{j\neq k}\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k}\mathrm{E}(\varepsilon_j^2)\mathrm{E}(\varepsilon_k^2) + \cdots = (\bar{B}^{(i)})^2 + \sigma^2\left[\bar{B}^{(i)}\sum_j\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j} + \sum_j\left(\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\right)^2\right] \\
&+ \sigma^4\left[\frac{3}{4}\sum_j\left(\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\right)^2 + \sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j^3}\bigg|_{\bar{T}_j} + \frac{\bar{B}^{(i)}}{4}\sum_{j\neq k}\frac{\partial^4 B^{(i)}}{\partial T_j^2\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} + \frac{\bar{B}^{(i)}}{4}\sum_j\frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\right. \\
&\left. + \frac{1}{4}\sum_{j\neq k}\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\frac{\partial^2 B^{(i)}}{\partial T_k^2}\bigg|_{\bar{T}_k} + \frac{1}{2}\sum_{j\neq k}\left(\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\right)^2 + \sum_{j\neq k}\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k}\right] + \mathrm{O}(\sigma^6) \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{E}(B^{(i)}))^2 =& (\bar{B}^{(i)})^2 + \sigma^2\bar{B}^{(i)}\sum_j\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j} + \sigma^4\left[\frac{\bar{B}^{(i)}}{4}\sum_{j\neq k}\frac{\partial^4 B^{(i)}}{\partial T_j^2\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k} + \frac{\bar{B}^{(i)}}{4}\sum_j\frac{\partial^4 B^{(i)}}{\partial T_j^4}\bigg|_{\bar{T}_j}\right. \\
&\left. + \frac{1}{4}\sum_{j\neq k}\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\frac{\partial^2 B^{(i)}}{\partial T_k^2}\bigg|_{\bar{T}_k} + \frac{1}{4}\sum_j\left(\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\right)^2\right] + \mathrm{O}(\sigma^6) \qquad (49)
\end{aligned}
$$

Substitute Eqs. (48) and (49) into (46), and retain terms of order $\mathrm{O}(\sigma^2)$ and $\mathrm{O}(\sigma^4)$.

$$
\begin{aligned}
\mathrm{Var}(B^{(i)}) =& \sigma^2\sum_j\left(\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\right)^2 + \sigma^4\left[\frac{1}{2}\sum_j\left(\frac{\partial^2 B^{(i)}}{\partial T_j^2}\bigg|_{\bar{T}_j}\right)^2 + \sum_j\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j^3}\bigg|_{\bar{T}_j} + \frac{1}{2}\sum_{j\neq k}\left(\frac{\partial^2 B^{(i)}}{\partial T_j\partial T_k}\bigg|_{\bar{T}_j,\bar{T}_k}\right)^2\right. \\
&\left. + \sum_{j\neq k}\frac{\partial B^{(i)}}{\partial T_j}\bigg|_{\bar{T}_j}\frac{\partial^3 B^{(i)}}{\partial T_j\partial T_k^2}\bigg|_{\bar{T}_j,\bar{T}_k}\right] \qquad (50)
\end{aligned}
$$

7. Results and discussion

Let the Biot number distribution be described by the following function:

$$B(t) = \begin{cases} 2t, & 0 \leq t < 0.5 \\ 2(1-t), & 0.5 \leq t < 0.75 \\ 0.5, & 0.75 \leq t \leq 1 \end{cases} \quad (51)$$

The direct problem, described by Eqs. (6)–(8) and (10), is solved by the explicit finite-difference method with uniform grid, $\Delta x = 0.01$, and $\Delta t = \Delta x^2/6$. This choice of Δt results in the solution that is accurate to fourth order in Δx. An inverse heat conduction problem can be constructed using Eqs. (6)–(9). The temperature measurements at $x_0 = 1$ in Eq. (9) are obtained from the solution to the direct problem. The inverse problem is then solved for Biot number components using the algorithm described above. The quality of the solution can readily be determined since the exact solution is known.

The quality of the estimation depends on $\Delta_n^{(i)}$ (nonlinear bias), $\mathrm{Var}(B^{(i)})$ (variance), and $\Delta_{d,B}$ (deterministic bias). The deterministic bias may be defined as

$$\Delta_{d,B} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(B(ip\Delta t) - \bar{B}^{(i)})^2} \quad (52)$$

where $B(t)$ is given by Eq. (51), and $\bar{B}^{(i)}$ is the expected value of estimated Biot number at time $ip\Delta t$ when the variance (σ^2) of input data is zero. There are three tunable parameters in the present method, n, r, and p. The effects of n on the quality of the solution are quite predictable. Hence, the number of n is set at 50, and only the effects of r and p on the solution will be considered.

Fig. 3 shows the variations of $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$ with i for $p = 1$, $r = 20$, and $\sigma^2 = 0.01$. In general, both nonlinear bias and variance vary from component to component. For this particular form of $B(t)$, both $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$ reach maximum at $i = n$. To compare results obtained with different p and r, it is sufficient to compare maximum $\Delta_n^{(i)}$ and $\mathrm{Var}(B^{(i)})$.

The future-time parameter r acts as a stabilizing parameter in the sequential function specification method. This is apparent from Table 1, where it is shown that increasing r, while keeping p constant, results in a more stable solution (lower maximum variance), but a less accurate one (higher deterministic bias). It is interesting to note that a more stable solution also has lower maximum nonlinear bias. When p is increased, and r is kept constant, Table 2 shows that variance, nonlinear bias, and deterministic bias all decrease. This means that, with the same number of Biot number components to be estimated, taking more measurements at one sensor location can lead to a more stable and accurate solution. Although the accuracy of the solution does not appear to improve much with p, the solution becomes noticeably more stable when p is increased. However, one should be cautioned that when p becomes too large, the time step for temperature measurements may be too small, causing correlation among different measurements, which will probably invalidate the above conclusion. Nevertheless, stabilizing the solution without deteriorating its accuracy by letting p equal to 2 or 3 is worth taking into consideration when designing an experiment since it is less costly and more convenient than increasing the number of sensors.

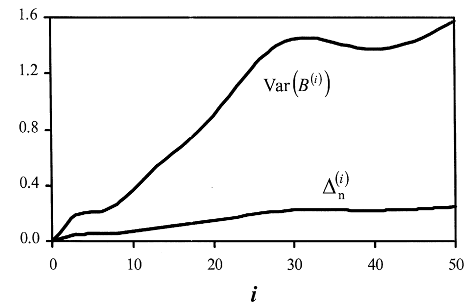

Fig. 3. Variations of variance and nonlinear bias of estimated Biot number components. Calculations were performed using $x_0 = 1.0$, $n = 50$, $r = 12$, $p = 1$, and $\sigma^2 = 0.01$.

In Fig. 4, three different plots of $E(B^{(i)})$ obtained with $\sigma^2 = 0$, 0.005, and 0.01 are compared with exact Biot number function. The parameters used in obtaining these results are $n = 50$, $r = 12$, $p = 1$, and $x_0 = 1$. It can be seen that, without taking nonlinear bias into consideration, the quality of estimated Biot number components is expected to worsen as the variance of temperature measurements increase. This is in contrast with the estimation of boundary heat flux or boundary temperature, where the expected value of the solution

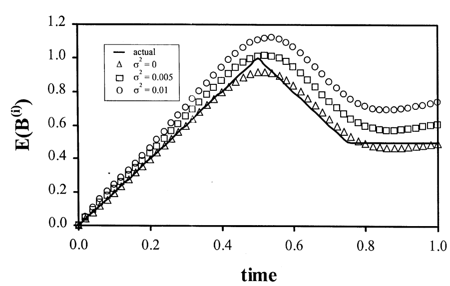

Fig. 4. Comparison between the expected values of estimated Biot number components at three different σ^2 and the exact Biot number distribution. Calculations were performed using $x_0 = 1.0$, $n = 50$, $r = 12$, and $p = 1$.

Table 1
Variations of maximum variance, maximum nonlinear bias, and deterministic bias with parameter r (n = 50, p = 1, x_0 = 1.0, σ^2 = 0.01)

r	$[\mathrm{Var}(B^{(i)})]_{\max}$	$[\Delta_n^{(i)}]_{\max}$	$\Delta_{d,B}$
10	7.02139	0.83771	0.01775
11	3.07827	0.43058	0.02202
12	1.57819	0.25020	0.02676
13	0.90504	0.15850	0.03178
14	0.56385	0.10705	0.03696
15	0.37418	0.07595	0.04232 1
16	0.26093	0.05603	0.04792 1
17	0.18936	0.04266	0.05381
18	0.14195	0.03334	0.06005
19	0.10929	0.02664	0.06663
20	0.08605	0.02168	0.07352

Table 2
Variations of maximum variance, maximum nonlinear bias, and deterministic bias with parameter p (n = 50, r = 12, x_0 = 1.0, σ^2 = 0.01)

p	$[\mathrm{Var}(B^{(i)})]_{\max}$	$[\Delta_n^{(i)}]_{\max}$	$\Delta_{d,B}$
1	1.57819	0.25020	0.02676
2	0.78634	0.13100	0.02573
3	0.51912	0.08846	0.02536
4	0.38661	0.06674	0.02518
5	0.30775	0.05356	0.02506

does not depend on the variance of temperature measurements.

Obviously, it is desirable to have as small σ^2 as possible. From the relation between T_i and T_i', one can see that the variance of actual temperature measurement T_i can be related to the variance of dimensionless temperature T_i' via the relation

$$\mathrm{Var}(T_i) = \sigma^2 = \frac{\mathrm{Var}(T'_i)}{(T_\infty - T_0)^2} \tag{53}$$

Thus, besides decreasing the variance of actual temperature measurement, increasing the difference between the ambient and the initial temperatures will also result in less variance in estimated $B^{(i)}$.

8. Conclusions

The solution to the one-dimensional inverse heat conduction problem of estimating time-dependent heat transfer coefficient has been presented. Estimations of boundary heat flux and boundary temperature are performed by using the sequential function specification method with piecewise linear basis functions and the assumption of linearly varying boundary heat flux or boundary temperature components. They are then used to obtain the solution for Biot number. It is found that, in addition to variance and deterministic bias, the solution is characterized by nonlinear bias, which results from the nonlinear dependence of the solution on measured temperatures. If certain statistical assumptions regarding the measurement errors are made, it has been shown that variance and nonlinear bias can be expressed as functions of variance of temperature measurements. For a given number of Biot number components to be estimated, the method of solution offers two tunable parameters. Whereas an increase in parameter r results in decreasing variance, decreasing nonlinear bias, and increasing deterministic bias, an increase in parameter p results in decreasing variance, decreasing nonlinear bias, and slightly decreasing deterministic bias.

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